TWO-DIMENSIONAL GENERALIZED GAMMA FUNCTION AND ITS APPLICATIONS

Summary. In this article, we present a new two-dimensional generalization of the gamma function based on the product of the one-dimensional generalized beta function and the one-dimensional generalized gamma function. As will become clear later, one of the properties of this generalization is a formula that generalizes the famous formula establishing the connection between the classical gamma and beta functions. Next, we present the properties of this generalization, some series for the generalized beta function and the generalized two-dimensional gamma function. As a practical application of the two-dimensional generalized gamma function, we will show how it can be used to represent a fairly wide class of double integrals in the form of functional series, or in the form of the product of two one-dimensional integrals, which makes it quite easy to find their values. That is, it will be seen how, with the help of simple transformations, many types of double integrals can be reduced to a generalized two-dimensional gamma function (23), which greatly simplifies work with them, thanks to formula (24). At the end of the article, a representation of the classical and generalized hypergeometric function in the form of a two-dimensional generalized gamma function is presented.

Keywords: special functions, generalization of the gamma function, generalization of the beta function, generalized formula for the relationship between the classical gamma function and the beta function.

Problem statement: finding a special case of a generalized two-dimensional gamma function, the property of which would be a formula that is a generalization of the well-known formula connecting the classical gamma and beta functions. Further, the statement of the problem is obviously to find the properties of a given generalized two-dimensional gamma function.

Analysis of recent research and publications. The properties and applications of classical special functions are well studied and are presented, for example, in [1; 2; 3; 4]. Among other things, this literature presents many properties of the classical special functions $B(\alpha, \beta)$ and $\Gamma(\alpha)$, their various generalizations. In this article we will consider a new two-dimensional generalization of the Gamma function. We will also obtain a generalization of the formula for the relationship between the classical gamma function and the beta function [3] to the case of any integrand functions.
Purpose statement (setting objectives). The statement of this article is to find the desired type of generalized two-dimensional gamma function that corresponds to the formulation of the problem.

The main research material. Throughout the article we will consider the parameters \( \alpha, \beta, \lambda, \nu, \omega, \gamma, a, b \) as real numbers only, with the exception of Remark 5 and the final application for hypergeometric functions at the end of the article. We will also assume throughout the entire article that \( R \) is the set of real numbers and \( N \) is the set of natural numbers.

1. Introduction of a two-dimensional generalization of the gamma function and study of its properties.

We will consider a one-dimensional generalization of the gamma function in the form

\[
\Gamma_{\gamma_1}(\omega) := \int_0^\infty g(x) x^{\omega-1} e^{-x} \, dx, \quad \omega > 0; \tag{1}
\]

one-dimensional generalization of the beta function in the form

\[
B_{\gamma_1}(\alpha, \beta) := \int_0^1 f(x) x^{\alpha-1} (1-x)^{\beta-1} \, dx, \quad \alpha > 0, \; \beta > 0; \tag{2}
\]

and two-dimensional generalization of the Gamma function

\[
\Gamma \left( R^2; \Omega(x, y) \right) := \int_{[x, y \in R^2, x > 0]} \Omega(x, y) x^{\nu-1} y^{\nu-1} e^{-x-y} \, dxdy, \tag{3}
\]

where \( \nu > 0, \; \omega > 0, \; \gamma > 0 \).

Obviously, if \( \Omega = 1 \) and \( \gamma = 0 \), then our generalized gamma function

\[
\Gamma \left( R^2; \Omega(x, y) \right) = \Gamma(\nu) \Gamma(\omega).
\]

In this article we will set the task of finding the product of generalizations (1) and (2) \( B_{\gamma_1}(\alpha, \beta) \Gamma_{\gamma_1}(\omega) \) in the form (3). That is, we will find for the function \( \Omega \) such a condition that this product looks like a special case of the right-hand side of the formula (3). Thus, we obtain a generalization of the well-known formula for the relationship between the classical gamma function and the beta function \( \Gamma(\alpha) \Gamma(\beta) = \beta(\alpha, \beta) \Gamma(\alpha + \beta) \).

To do this, we will consider the following generalizations of the gamma function of the form

\[
\Gamma_{\gamma_1}(\alpha + \beta + \lambda) := \int_0^\infty g(x) x^{\lambda-1} e^{-x} \, dx, \quad \alpha > 0, \; \beta > 0, \; \lambda > 0;
\]

and provided that integrals (4), (2) exist, we will obtain a double integral, which will be equal to their product. We will also consider integrals

\[
I \left( R^2; f \left( \frac{x+y}{x+y} \right) g \left( \frac{\nu + \sigma}{\nu + \sigma} \right) \right) (\alpha, \beta, \lambda) := \int_{[x, y \in R^2, x > 0]} f \left( \frac{y}{x+y} \right) g(x+y). \tag{5}
\]

\[
I^{(1)} \left( R^2; f \left( \frac{x+y}{x+y} \right) g \left( \frac{\nu + \sigma}{\nu + \sigma} \right) \right) (\alpha, \beta, \lambda) := \int_{[x, y \in R^2, x > 0]} f \left( \frac{y}{x+y} \right) g(x+y) y^{\nu-1} e^{-x-y} \, dxdy; \tag{6}
\]

\[
I^{(2)} \left( R^2; f \left( \frac{x+y}{x+y} \right) g \left( \frac{\nu + \sigma}{\nu + \sigma} \right) \right) (\alpha, \beta, \lambda) := \int_{[x, y \in R^2, x > 0]} f \left( \frac{y}{x+y} \right) g(x+y) y^{\nu-1} e^{-x-y} \, dxdy. \tag{7}
\]

where \( \alpha > 0, \beta > 0, \lambda > 0 \) for all three previous integrals.

We will see later in our study that in the generalization (3) it is convenient to choose the order of variables \( \Omega(x, y) \), and not \( \Omega(x, y) \).

Let us introduce the following notation

\[
Q(x, y) := f \left( \frac{y}{x+y} \right) g(x+y). \tag{8}
\]

**Theorem 1.** Let us assume that \( Q(x, y) \) is a given function satisfying the following conditions:

1) the function \( Q(x, y) \) exists, is non-negative, continuous at all points of the region \( T = \{(x, y) \in R^2; x > 0, y > 0\} \), with the possible exception of a finite number of points \( M_1, M_2, ..., M_\nu \) in this region. At points \( M_1, M_2, ..., M_\nu \) the function \( Q(x, y) \) tends to \( +\infty \);

2) integrals (4), (2), (5), (6), (7) exist and converge for all values \( \alpha > 0, \beta > 0, \lambda > 0 \);

3) the following equality holds

\[
I \left( R^2; f \left( \frac{x+y}{x+y} \right) g \left( \frac{\nu + \sigma}{\nu + \sigma} \right) \right) (\alpha, \beta, \lambda) = I^{(1)} \left( R^2; f \left( \frac{x+y}{x+y} \right) g \left( \frac{\nu + \sigma}{\nu + \sigma} \right) \right) (\alpha, \beta, \lambda) =
\]

\[
= I^{(2)} \left( R^2; f \left( \frac{x+y}{x+y} \right) g \left( \frac{\nu + \sigma}{\nu + \sigma} \right) \right) (\alpha, \beta, \lambda), \quad \alpha > 0, \beta > 0, \lambda > 0. \tag{9}
\]
Then the following formula will be valid

\[
B_{f,\gamma}(\alpha,\beta)\Gamma_{\gamma,\delta}(\alpha+\beta+\lambda) = \int_0^\infty R^2 \cdot f \left( \frac{\beta}{\gamma + \beta} \right) g \left( \frac{\theta}{\gamma + \theta} \right) (\alpha,\beta;\lambda),
\]

where \( \alpha > 0, \beta > 0, \lambda > 0 \).

Proof.

\[
B_{f,\gamma}(\alpha,\beta) = \int_0^\infty f(x) x^{\alpha-1} (1-x)^{\beta-1} dx = \int_0^\infty \left( \frac{\Gamma(x+1)}{\Gamma(x)} \right) x^{\alpha-1} (1-x)^{\beta-1} dx =
\]

\[
= \int_0^\infty \left( \int_0^1 \tau^\alpha e^{-\tau} \tau^{\beta-1} dt \right) x^{\alpha-1} (1-x)^{\beta-1} dx = 2 \int_0^\infty \left( \int_0^1 \tau^\alpha e^{-\tau} \tau^{\beta-1} dt \right) x^{2\alpha-1} (1-x)^{\beta-1} dx =
\]

\[
= 2 \int_0^\infty \left( \int_0^\infty \sin^{2\alpha-1}(\phi) \cos^{2\beta-1}(\phi) d\phi \right) \sin^{2\alpha-1}(\phi) \cos^{2\beta-1}(\phi) d\phi.
\]

(11)

Let's multiply both sides of formula (11) by \( r^{2\alpha+2\beta-1} e^{-r^2} g(r^2) \) and integrate it's from \( 0 \) to \( +\infty \). Then we obtain

\[
B_{f,\gamma}(\alpha,\beta) \int_0^\infty (r^2)^{2\alpha+2\beta-1} e^{-r^2} dr =
\]

\[
= 2 \int_0^\infty \left( \int_0^\infty \sin^{2\alpha-1}(\phi) \cos^{2\beta-1}(\phi) d\phi \right) \int_0^\infty (r^2)^{2\alpha+2\beta-1} e^{-r^2} r^2 dr d\phi =
\]

\[
= 2 \int_0^\infty \left( \int_0^\infty \sin^{2\alpha-1}(\phi) \cos^{2\beta-1}(\phi) d\phi \right) \int_0^\infty (r^2)^{2\alpha+2\beta-1} e^{-r^2} r^2 dr d\phi =
\]

\[
= 2 \int_0^\infty \left( \int_0^\infty \sin^{2\alpha-1}(\phi) \cos^{2\beta-1}(\phi) d\phi \right) \int_0^\infty (x^2 + y^2)^{2\alpha+2\beta-1} (x^2 + y^2)^{2\alpha+2\beta-1} e^{-r^2} dx dy =
\]

\[
= 2 \int_0^\infty \left( \int_0^\infty \sin^{2\alpha-1}(\phi) \cos^{2\beta-1}(\phi) d\phi \right) \int_0^\infty (x^2 + y^2)^{2\alpha+2\beta-1} (x^2 + y^2)^{2\alpha+2\beta-1} e^{-r^2} dr dx =
\]

\[
= 2 \int_0^\infty \left( \int_0^\infty \sin^{2\alpha-1}(\phi) \cos^{2\beta-1}(\phi) d\phi \right) \int_0^\infty (x^2 + y^2)^{2\alpha+2\beta-1} (x^2 + y^2)^{2\alpha+2\beta-1} e^{-r^2} dr dx =
\]

\[
= 2 \int_0^\infty \left( \int_0^\infty \sin^{2\alpha-1}(\phi) \cos^{2\beta-1}(\phi) d\phi \right) \int_0^\infty (x^2 + y^2)^{2\alpha+2\beta-1} (x^2 + y^2)^{2\alpha+2\beta-1} e^{-r^2} dr dx =
\]

\[
= 2 \int_0^\infty \left( \int_0^\infty \sin^{2\alpha-1}(\phi) \cos^{2\beta-1}(\phi) d\phi \right) \int_0^\infty (x^2 + y^2)^{2\alpha+2\beta-1} (x^2 + y^2)^{2\alpha+2\beta-1} e^{-r^2} dr dx =
\]

\[
= 2 \int_0^\infty \left( \int_0^\infty \sin^{2\alpha-1}(\phi) \cos^{2\beta-1}(\phi) d\phi \right) \int_0^\infty (x^2 + y^2)^{2\alpha+2\beta-1} (x^2 + y^2)^{2\alpha+2\beta-1} e^{-r^2} dr dx =
\]

\[
= 2 \int_0^\infty \left( \int_0^\infty \sin^{2\alpha-1}(\phi) \cos^{2\beta-1}(\phi) d\phi \right) \int_0^\infty (x^2 + y^2)^{2\alpha+2\beta-1} (x^2 + y^2)^{2\alpha+2\beta-1} e^{-r^2} dr dx =
\]

\[
= 2 \int_0^\infty \left( \int_0^\infty \sin^{2\alpha-1}(\phi) \cos^{2\beta-1}(\phi) d\phi \right) \int_0^\infty (x^2 + y^2)^{2\alpha+2\beta-1} (x^2 + y^2)^{2\alpha+2\beta-1} e^{-r^2} dr dx =
\]
$$= \frac{1}{2} \int_0^\infty \int_0^\infty g(x + y) y^{\alpha+1} (x + y)^{\beta+1} e^{-\beta y} \, dy \, dx.$$ 

Since
$$\int_0^\infty y^{\alpha+1} (x + y)^{\beta+1} e^{-\beta y} \, dy = \frac{\Gamma(\alpha + \beta + \lambda)}{\Gamma(\alpha + 1)}$$
we obtain
$$\frac{1}{2} \Gamma_{1,1}(\alpha + \beta + \lambda) = \frac{1}{2} \int_0^\infty \int_0^\infty g(x + y) y^{\alpha+1} (x + y)^{\beta+1} e^{-\beta y} \, dy \, dx.$$ 

Next, we will obtain a similar theorem for the case of an alternating function $Q$. For this we will consider the integrals
$$\Gamma_{1,1}(\alpha + \beta + \lambda) = \int_0^\infty \int_0^\infty g(x + y) y^{\alpha+1} (x + y)^{\beta+1} e^{-\beta y} \, dy \, dx, \quad \alpha > 0, \quad \beta > 0, \quad \lambda > 0;$$
(12)
$$B_{1,1}(\alpha, \beta) = \int_0^\infty \int_0^\infty f(x) x^{\alpha-1} (1 - x)^{\beta-1} \, dx \, dy, \quad \alpha > 0, \quad \beta > 0;$$
(13)
$$f_1 \left[ \int_0^\infty \int_0^\infty f \left( \frac{y}{x} + \alpha \right) g \left( \frac{y}{x} + \beta \right) \right] (x, y) \, dx \, dy = \frac{1}{2} \int_0^\infty \int_0^\infty f(x) x^{\alpha-1} (x + y)^{\beta+1} e^{-\beta y} \, dy \, dx;$$
(14)
$$f_2 \left[ \int_0^\infty \int_0^\infty f \left( \frac{y}{x} + \alpha \right) g \left( \frac{y}{x} + \beta \right) \right] (x, y) \, dx \, dy = \frac{1}{2} \int_0^\infty \int_0^\infty f(x) x^{\alpha-1} (x + y)^{\beta+1} e^{-\beta y} \, dy \, dx,$$
(15)
and
$$f_3 \left[ \int_0^\infty \int_0^\infty f \left( \frac{y}{x} + \alpha \right) g \left( \frac{y}{x} + \beta \right) \right] (x, y) \, dx \, dy = \frac{1}{2} \int_0^\infty \int_0^\infty f(x) x^{\alpha-1} (x + y)^{\beta+1} e^{-\beta y} \, dy \, dx,$$
(16)
where $\alpha > 0, \beta > 0, \lambda > 0$ for all three previous integrals.

**Theorem 2.** Let us assume that $Q(x, y) = f \left( \frac{y}{x + y} \right) g(x + y)$ is a given function satisfying the following conditions:

1) the function $Q(x, y)$ exists and is continuous at all points of the region $T = \{(x, y) \in R^2 : x > 0, y > 0\}$, with the possible exception of a finite number of points $M_1, M_2, \ldots, M_n$ in this region.

2) the function $Q(x, y)$ tends to $+\infty$ or to $-\infty$ at points $M_1, M_2, \ldots, M_n$.

3) the equality (9) holds.

Then the formula (10) will be valid.

**Remark 1.** Theorems 1 and 2 will be valid if in condition 1) we replace the continuity of function $Q(x, y)$ in the region $T = \{(x, y) \in R^2 : x > 0, y > 0\}$ with the exception of a finite number of points $M_1, M_2, \ldots, M_n$, with continuity of function $Q(x, y)$ in the region $T = \{(x, y) \in R^2 : x > 0, y > 0\}$, with the exception of a finite number of piecewise smooth curves.

Next, we will consider the case of functions that are Riemann integrable.

**Theorem 3.** Let us assume that $Q(x, y) = f \left( \frac{y}{x + y} \right) g(x + y)$ is a given function satisfying the following conditions: the functions
$$H(x, y; \alpha, \beta, \lambda) := f \left( \frac{y}{x + y} \right) g(x + y) y^{\alpha-1} (x + y)^{\beta+1} e^{-\beta y},$$
(17)
$$\alpha > 0, \quad \beta > 0, \quad \lambda > 0;$$
$$H_2(x, y; \alpha, \beta, \lambda) := f \left( \frac{1}{x + y} \right) g \left( \frac{1}{x + y} \right) \frac{1}{x + y} \frac{1}{x + y} e^{-\beta y},$$
(18)
$$\alpha > 0, \quad \beta > 0, \quad \lambda > 0,$$
be Riemann integrable functions in the region $T_i = \{(x, y) \in R^2 : x > 0, y > 0, x \leq 1, y \leq 1\}$.

Then the formula (10) will be valid.

**Proof.** The necessity of the conditions of the theorem here is obvious.

$$f_3 \left[ \int_0^\infty \int_0^\infty f \left( \frac{y}{x + y} \right) g \left( \frac{y}{x + y} \right) \right] (x, y) \, dx \, dy =$$
$$= \frac{1}{2} \int_0^\infty \int_0^\infty f(x) x^{\alpha-1} (x + y)^{\beta+1} e^{-\beta y} \, dy \, dx +$$
$$+ \frac{1}{2} \int_0^\infty \int_0^\infty f(x) x^{\alpha-1} (x + y)^{\beta+1} e^{-\beta y} \, dy \, dx.$$
Here we should show that double integrals, when changing the order of integration, will give the same result.

\[
\int_0^\infty \left( \int_0^\infty \Phi(y+y) \right) g(y+y) d\gamma (x+y) e^{x+y} dy dx =
\]

\[
= \int_0^\infty \left( \int_0^\infty \Phi(x+y) \right) g(x+y) y^{x+y} (x+y) e^{x+y} dx dy =
\]

\[
= \int_0^\infty \left( \int_0^\infty \Phi(y+y) \right) g(y+y) y^{y+y} (y+y) e^{y+y} dy dx =
\]

\[
= \int_0^\infty \left( \int_0^\infty \Phi(y+y) \right) g(y+y) y^{y+y} (y+y) e^{y+y} dx dy =
\]

Next, we get

\[
\int_0^\infty \left( \int_0^\infty \Phi(x+y) \right) g(x+y) y^{x+y} (x+y) e^{x+y} dy dx =
\]

\[
= \int_0^\infty \left( \int_0^\infty \Phi(y+y) \right) g(y+y) y^{y+y} (y+y) e^{y+y} dx =
\]

\[
= \Gamma_{\alpha, \beta, \lambda} \int_0^\infty \left( \int_0^\infty \Phi(x+y) \right) g(x+y) y^{x+y} (x+y) e^{x+y} dx =
\]

Substitution (20) into (19), we obtain formula (10).

Next, changing the order of integration we obtain

\[
\int_0^\infty \left( \int_0^\infty \Phi(x+y) \right) g(x+y) y^{x+y} (x+y) e^{x+y} dx dy =
\]

\[
= \int_0^\infty \left( \int_0^\infty \Phi(y+y) \right) g(y+y) y^{y+y} (y+y) e^{y+y} dx =
\]

\[
= \Gamma_{\alpha, \beta, \lambda} \int_0^\infty \left( \int_0^\infty \Phi(x+y) \right) g(x+y) y^{x+y} (x+y) e^{x+y} dx =
\]

Substitution (22) into (21), we obtain formula (9). Thus, under the assumptions of this theorem, an integral of the form (6) does not depend on the order of integration. 

Next, we will consider the case of a continuous function \(Q\).

**Theorem 4.** Let us assume that \(Q(x, y) = \int \frac{y}{x+y} g(x+y)\) is a given function satisfying the following conditions:

1) the function \(Q(x, y)\) exists, is non-negative and is continuous at all points of the region \(T = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}\);

2) integrals (4), (2), (5), (6), (7) exist and converge for all values \(0 > \alpha, \beta > 0, \lambda > 0\);

3) the equality (9) holds.

Then the formula (10) will be valid.

**Proof.** Obviously, Theorem 4 is a special case of Theorem 1.

**Remark 2.** Theorems 1, 2, 4 and Remark 1 will be valid if in condition 2) of Theorems 1, 2, 4 we require the existence of only integrals (4), (2) or the existence only integrals (5), (6), (7).

Next, we will introduce the definition of the generalized two-dimensional gamma function in the form of a double integral. One of the main properties of this function is a generalization of the well-known formula for the relationship between the classical gamma function and the beta function.

**Definition 1.** Let the function \(Q(x, y) = \int \frac{y}{x+y} g(x+y)\) satisfy the conditions of at least one of Theorems 1-4. We define the two-dimensional generalized gamma function as a double integral as follows:

\[
\Gamma \left[ \mathbb{R}^2 : \int \frac{y}{x+y} g(x+y) \right] (x, y) = \int_0^\infty \int_0^\infty \frac{y}{x+y} g(x+y) \left( y^{x+y} (x+y) e^{x+y} \right) dx dy,
\]

\(\alpha > 0, \beta > 0, \lambda > 0\).

(23)
Theorem 5 (Properties of the two-dimensional generalized gamma function). We assume that the conditions of at least one of Theorems 1-4 are satisfied. Then for values of $\alpha > 0$, $\beta > 0$, $\lambda \geq 0$, we obtain the following properties of the two-dimensional generalized gamma function:

1.) $\Gamma \left( \mathcal{R}^2; f \left( \frac{x}{y} + \mathbf{a} \right) g \left( y + \mathbf{b} \right) \right)(\alpha, \beta; \lambda) = \Gamma \left( \mathcal{R}^2; f \left( 1 - \frac{x}{y} + \mathbf{a} \right) g \left( y + \mathbf{b} \right) \right)(\alpha, \beta; \lambda);$

2.) $\Gamma \left( \mathcal{R}^2; f \left( \frac{x}{y} + \mathbf{a} \right) g \left( y + \mathbf{b} \right) \right)(\alpha, \beta; \lambda) = \int \int_{\mathcal{R}^2} \frac{f(x,y)}{(x+y)^{\alpha-1}(y+x)^{\beta-1}} e^{-\frac{x}{y}} dx dy =\int \int_{\mathcal{R}^2} \frac{f(x,y)}{(x+y)^{\alpha-1}(y+x)^{\beta-1}} e^{-\frac{x}{y}} dx dy ;$

3.) $\Gamma \left( \mathcal{R}^2; f \left( \frac{x}{y} + \mathbf{a} \right) g \left( y + \mathbf{b} \right) \right)(\alpha, \beta; \lambda) = \int \int_{\mathcal{R}^2} \frac{f(x,y)}{(x+y)^{\alpha-1}(y+x)^{\beta-1}} e^{-\frac{x}{y}} dx dy ;$

4.) $\Gamma \left( \mathcal{R}^2; f \left( \frac{x}{y} + \mathbf{a} \right) g \left( y + \mathbf{b} \right) \right)(\alpha, \beta; \lambda) = \int \int_{\mathcal{R}^2} \frac{f(x,y)}{(x+y)^{\alpha-1}(y+x)^{\beta-1}} e^{-\frac{x}{y}} dx dy ;$

5.) Generalized formula for the relationship between the gamma function and the beta function

\[ B_{\mathcal{R}^2}(\alpha, \beta) \Gamma_{\mathcal{R}^2}(\alpha + \beta + \lambda) = \Gamma \left( \mathcal{R}^2; f \left( \frac{x}{y} + \mathbf{a} \right) g \left( y + \mathbf{b} \right) \right)(\alpha, \beta; \lambda); \]  

5.1. If we assume $f = 1$, $g = 1$, $\lambda = 0$ in the formula (24), then we obtain formula for the relationship between the classical gamma function and the beta function

\[ B(\alpha, \beta) \Gamma(\alpha + \beta) = \Gamma(\alpha) \Gamma(\beta); \]  

5.2. If we assume $f = 1$, $g = 1$, $\alpha = 1$, $\beta = 1$ in the formula (24), then we obtain classical gamma function

\[ \Gamma \left( \mathcal{R}^2; (1,1) \right)(1,1) = \Gamma(2 + \lambda); \]  

5.3. If we assume $f = 1$, $g = 1$, $\alpha = 1/2$, $\beta = 1/2$ in the formula (24), then we also find a special case of transforming the two-dimensional generalized gamma function into the classical gamma function

\[ \Gamma \left( \mathcal{R}^2; \left(1/2,1/2,1 \right) \right) = \pi \Gamma(1 + \lambda); \]  

5.4. If we assume $f = 1$, $g = 1$ in the formula (24), then we obtain

\[ \int_0^\infty x^{\alpha-1} e^{-x} dx = \Gamma(\alpha + \beta + \lambda) B(\alpha, \beta); \]  

6. If the function $g$ identically not equal to zero, then

\[ \Gamma \left( \mathcal{R}^2; f \left( \frac{x}{y} + \mathbf{a} \right) g \left( y + \mathbf{b} \right) \right)(\alpha, \beta; \lambda) \Gamma \left( \mathcal{R}^2; f \left( \frac{x}{y} + \mathbf{a} \right) g \left( y + \mathbf{b} \right) \right)(\alpha, \beta; \lambda + 1) = \frac{\Gamma \left( \mathcal{R}^2; f \left( \frac{x}{y} + \mathbf{a} \right) g \left( y + \mathbf{b} \right) \right)(\alpha, \beta; \lambda + 1)}{\Gamma \left( \mathcal{R}^2; f \left( \frac{x}{y} + \mathbf{a} \right) g \left( y + \mathbf{b} \right) \right)(\alpha, \beta; \lambda + 1)}; \]  

7. If the limit is valid $\lim_{x \to 0} e^{-x^\alpha} g(x)^{\beta+\lambda} = 0$, if $x \to 0$ and $x \to \infty$, then we find

\[ \Gamma \left( \mathcal{R}^2; f \left( \frac{x}{y} + \mathbf{a} \right) g \left( y + \mathbf{b} \right) \right)(\alpha, \beta; \lambda + 1) = (\alpha + \beta + \lambda) \Gamma \left( \mathcal{R}^2; f \left( \frac{x}{y} + \mathbf{a} \right) g \left( y + \mathbf{b} \right) \right)(\alpha, \beta; \lambda + 1) + \Gamma \left( \mathcal{R}^2; f \left( \frac{x}{y} + \mathbf{a} \right) g \left( y + \mathbf{b} \right) \right)(\alpha, \beta; \lambda)\].

Proof. Properties 1-4 are obvious. Property 5 follows from Definition 1 of the two-dimensional generalized gamma function (23) and the formula (10), due to the fulfillment of the conditions of at least one of Theorems 1-4. Property 7 follows from Property 5 and the following formula, which is satisfied under the same assumptions

\[ \Gamma_{\mathcal{R}^2}(\alpha + \beta + \lambda + 1) = (\alpha + \beta + \lambda) \Gamma_{\mathcal{R}^2}(\alpha + \beta + \lambda) + \Gamma_{\mathcal{R}^2}(\alpha + \beta + \lambda + 1), \]

$\alpha > 0$, $\beta > 0$, $\lambda \geq 0$. ✠
Lemma 1. Let us assume that following conditions are met:
1) the function \( f'(x) \) exists on \((0,1)\) and the following integral exists
\[
B_{j/(i)}(\alpha + 1, \beta + 1) = \int_0^1 f'(x)x^\alpha(1-x)^\beta \, dx , \quad \alpha > 0, \beta > 0 ;
\]
(29)
2) the following limits are valid
\[
\lim_{x \to 0^+} x^\alpha(1-x)^\beta = 0, \text{ if } x \to 0^+ \text{ and } x \to 1^- , \alpha > 0, \beta > 0.
\]
Then the following formula will be valid
\[
B_{j/(i)}(\alpha, \beta) - B_{j/(i)}(\alpha + 1, \beta) - B_{j/(i)}(\alpha, \beta + 1) = 0, \quad \alpha > 0, \beta > 0.
\]
(30)
Proof. We obtain
\[
B_{j/(i)}(\alpha + 1, \beta + 1) = \int_0^1 f'(x)x^\alpha(1-x)^\beta \, dx = -\alpha B_{j/(i)}(\alpha, \beta + 1) + \beta B_{j/(i)}(\alpha + 1, \beta).
\]
(31)
Next, we get
\[
\int_0^1 f'(x)x^\alpha(1-x)^\beta \, dx = \int_0^1 f(x)(\alpha x^{\alpha-1}(1-x)^\beta - \beta x^{\alpha}(1-x)^{\beta-1}) \, dx =
\]
\[
= -\alpha B_{j/(i)}(\alpha, \beta) + (\alpha + \beta) B_{j/(i)}(\alpha + 1, \beta) , \quad \alpha > 0, \beta > 0.
\]
(32)
Equating the right sides of formulas (31) and (32), we obtain (30). \( \diamond \)

Lemma 2. Let us assume that following conditions are met:
1) conditions 1) and 2) of Lemma 1 are satisfied;
2) function \( Q(x, y) = \int \frac{y}{x+y} g(x+y) \) satisfy the conditions of at least one of Theorems 1-4.
Then the following formulas will be valid
\[
\Gamma \left\{ R^2_3; f \left( \frac{y}{x+y} \right) g(x+y) \right\} (\alpha + 1, \beta + 1, \lambda) =
\]
\[
= \Gamma \left\{ R^2_3; f \left( \frac{y}{x+y} \right) g(x+y) \right\} (\alpha + 1, \beta + 1, \lambda) , \quad \alpha > 0, \beta > 0, \lambda > 0 ;
\]
(33)
\[
\beta \Gamma \left\{ R^2_3; f \left( \frac{y}{x+y} \right) g(x+y) \right\} (\alpha + 1, \beta + 1, \lambda) - \alpha \Gamma \left\{ R^2_3; f \left( \frac{y}{x+y} \right) g(x+y) \right\} (\alpha + 1, \beta + 1, \lambda) =
\]
\[
= \Gamma \left\{ R^2_3; f \left( \frac{y}{x+y} \right) g(x+y) \right\} (\alpha + 1, \beta + 1, \lambda) , \quad \alpha > 0, \beta > 0, \lambda > 0 .
\]
(34)
Proof. By multiplying equality (30) by \( \Gamma_{\alpha + 1} (\alpha + \beta + \lambda + 1) \), and then based on the definition of the two-dimensional generalized gamma function, we find (31).

We obtain formula (34) using a similar way from formula (31). \( \diamond \)

Lemma 3. Let’s the function \( Q(x, y) = \int \frac{y}{x+y} g(x+y) \) satisfy the conditions of at least one of Theorems 1-4. Then for \( \forall \, \alpha > 0, \beta > 0, \lambda > 0 \) and \( \forall \, l, m, n \), where \( \{ l, m, n \} \subset N \cup \{ 0 \} \), exist derivatives of all orders for the two-dimensional generalized gamma function
\[
\Gamma \left\{ R^2_3; f \left( \frac{y}{x+y} \right) g(x+y) \right\} (\alpha, \beta; \lambda) \text{ and the following formula is valid}
\]
\[
\frac{\partial^l}{\partial \lambda^l} \frac{\partial^m}{\partial \alpha^m} \frac{\partial^n}{\partial \beta^n} \Gamma \left\{ R^2_3; f \left( \frac{y}{x+y} \right) g(x+y) \right\} (\alpha, \beta; \lambda) =
\]
\[
= \int \int \left( \frac{y}{x+y} \right) g(x+y) (\log(x+y))^l (\log y)^m y^{\alpha-1} x^{\beta-1} (x+y)^\lambda e^{-x-y} \, dx \, dy ,
\]
where \( \alpha > 0, \beta > 0, \lambda > 0, \{ l, m, n \} \subset N \cup \{ 0 \} .
\]

Theorem 6 (Formula for differentiating the two-dimensional generalized gamma function). Let’s the function \( Q(x, y) = \int \frac{y}{x+y} g(x+y) \) satisfy the conditions of at least one of Theorems 1-4. Then the following formula will be valid
\[
\int \int \frac{y}{x+y} g(x+y) (\log(x+y))^l (\log y)^m y^{\alpha-1} x^{\beta-1} (x+y)^\lambda e^{-x-y} \, dx \, dy =
\]
\[
= \sum \sum \sum \sum \int \int g(r) (\log r)^{l-1} r^{m-1} e^{-r} dr.
\]
\[
\int_{0}^{1} f(x)(\log x)^{\alpha+\beta} (1-x)^{\gamma-1}(1-x)^{\beta+1}dx,
\]
where \( \alpha > 0, \beta > 0, \gamma > 0, \{l,m,n\} \subseteq N \cup \{0\}, \left\{ \frac{L}{M} \right\} = \frac{L!}{M!(L-M)!}.

In the notation style, according to Definition 1, our differentiating formula (35) has the form

\[
\frac{\partial^n}{\partial x^n} \frac{\partial^n}{\partial y^n} \frac{\partial^n}{\partial z^n} \Gamma \left[ \left( \int_{0}^{\infty} f \left( \frac{y}{x+y} \right) g(x+y) \left( \log(x+y) \right)^{\alpha} (\log(y))^\beta (\log(x))^\gamma \right) dx \right] =
\]

\[
= \sum_{i+j+k=n} \binom{n}{i,j,k} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \left( \frac{y}{x+y} \right) g(x+y) \left( \log(x+y) \right)^{\alpha} (\log(y))^\beta (\log(x))^\gamma \]

where \( \alpha > 0, \beta > 0, \gamma > 0, \{l,m,n\} \subseteq N \cup \{0\}.

Proof.

\[
\frac{\partial^n}{\partial x^n} \frac{\partial^n}{\partial y^n} \frac{\partial^n}{\partial z^n} \Gamma \left[ \left( \int_{0}^{\infty} f \left( \frac{y}{x+y} \right) g(x+y) \left( \log(x+y) \right)^{\alpha} (\log(y))^\beta (\log(x))^\gamma \right) dx \right] =
\]

\[
= \sum_{i+j+k=n} \binom{n}{i,j,k} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \left( \frac{y}{x+y} \right) g(x+y) \left( \log(x+y) \right)^{\alpha} (\log(y))^\beta (\log(x))^\gamma \]

\[
= \sum_{i+j+k=n} \binom{n}{i,j,k} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \left( \frac{y}{x+y} \right) g(x+y) \left( \log(x+y) \right)^{\alpha} (\log(y))^\beta (\log(x))^\gamma \]

In this proof we applied the transformations.
\[
\frac{y^2}{x^2 + y^2} = \frac{\frac{y^2}{x^2 + y^2} \cdot e^{-y}}{\Gamma(\frac{y^2}{x^2 + y^2} \cdot e^{-y}) dt} = \int_0^\infty \frac{\Gamma(\frac{y^2}{x^2 + y^2} \cdot e^{-y})}{\Gamma(\frac{y^2}{x^2 + y^2} \cdot e^{-y}) dt} \left( \frac{y^2}{x^2 + y^2} \right)^{x^2 + y^2} \cdot e^{-y} dt,
\]

where \( a > 0, \lambda > 0, \{m,n\} \subset N \cup \{0\} \).

Remark 3. Using the equality \( B_{\gamma j}(\alpha, \beta) = B_{\gamma j}^{-1}(\beta, \alpha) \), from formula (36), we find

\[
\int_0^\infty \int_0^\infty f \left( \frac{x}{x+y} \right) g(x+y) \left( \log(x+y) \right)^m \left( \log y \right)^n y^{x-1} x^{y-1} (x+y)^{\gamma-1} \cdot e^{-\gamma (x+y)} dydx = \sum_{j=0}^{\infty} \sum_{i=0}^{m} \binom{m}{j} \Gamma((i+j)(\alpha + \beta + \lambda)) \frac{\partial^{i+j}}{\partial \alpha^i \partial \beta^j} B_{\gamma j}(\alpha, \beta),
\]

where \( \alpha > 0, \beta > 0, \lambda > 0, \{m,n\} \subset N \cup \{0\} \). The formula (37) is also a generalization of the relation between the classical gamma function and the beta function \( \Gamma(\alpha \Gamma(\beta) = \Gamma(\alpha + \beta)B(\alpha, \beta) \), if \( m = n = 0 \).

Remark 5. We will assume that \( \gamma_0, \gamma_0 \in R \), then there are all these values belonging to the set of real numbers. Theorems 1-6, Lems 1-3 and Remark 1-4 will be valid if we consider our values of \( \alpha, \beta, \lambda \) in the complex plane, provided that \( \alpha = \gamma_0, \beta = \gamma_0 \geq 0, \lambda = \gamma_0 \geq 0 \).

2. Practical applications of the generalized two-dimensional gamma function for calculating double integrals, including those containing some special functions.

In this chapter, we will present several theorems containing integral equalities using the two-dimensional generalized gamma function.

Theorem 7. Let's the function \( f , g = 1 \) satisfy the conditions of at least one of Theorems 1-4, provided that \( g = 1 \). Then for any \( a > 0, \alpha > 0, \lambda > 0 \) the following formulas will be valid:

\[
\int_0^\infty \int_0^\infty f \left( \frac{y}{x+y} \right) g(x+y) \left( \log(x+y) \right)^m \left( \log y \right)^n y^{x-1} x^{y-1} (x+y)^{\gamma-1} \cdot e^{-\gamma (x+y)} dydx = \Gamma(\beta + \lambda) \left( x + \lambda \arctan \frac{y}{\lambda} \right) B_{\gamma j}(\alpha, \beta),
\]

where \( \alpha > 0, \beta > 0, \lambda > 0, \{m,n\} \subset N \cup \{0\} \).

Proof. We obtain

\[
B_{\gamma j}(\alpha, \beta) = \int_0^{2\pi} e^{-\alpha \cos \theta + \beta \sin \theta} d\theta = \frac{\gamma j \cdot \Gamma(\gamma j) \cdot \Gamma(\beta + \lambda) \left( x + \lambda \arctan \frac{y}{\lambda} \right) B_{\gamma j}(\alpha, \beta)}{\gamma j \cdot \Gamma(\gamma j) \cdot \Gamma(\beta + \lambda) \left( x + \lambda \arctan \frac{y}{\lambda} \right) B_{\gamma j}(\alpha, \beta),
\]

where \( \alpha > 0, \beta > 0, \lambda > 0, \{m,n\} \subset N \cup \{0\} \).
And further, we find
\[ B_{\eta_1}(\alpha, \beta) \int_0^r t^{2n+2\eta+2+\lambda} e^{-(\alpha + \lambda \eta)t} \, dt = \]
\[ = B_{\eta_1}(\alpha, \beta) \int_0^r t^{2n-2\beta_0-2+\lambda} e^{-\alpha t} \cos (b r^2) \, dr - i \frac{1}{2} B_{\eta_1}(\alpha, \beta) \int_0^r t^{2n+2\eta+2+\lambda} e^{\alpha t} \sin (b r^2) \, dr = \]
\[ = \frac{1}{2} B_{\eta_1}(\alpha, \beta) \int_0^r t^{2n+2\eta+2+\lambda} e^{-\alpha t} \cos (b r^2) \, dr - \frac{1}{2} B_{\eta_1}(\alpha, \beta) \int_0^r t^{2n+2\eta+2+\lambda} e^{\alpha t} \sin (b r^2) \, dr. \] (41)

Equating the real and imaginary parts of equalities (40), (41), we obtain (38), (39), with further consideration of the following two equalities at \( \alpha > 0, \beta > 0, \lambda \geq 0 : \)
\[ \int_0^r r^{2n+2\eta+1-1} e^{-\alpha t} \cos (b r^2) \, dt = \frac{\Gamma(\alpha + \beta + \lambda)}{(\alpha^2 + b^2)^{\eta+1}} \cos \left( (\alpha + \beta + \lambda) \arctan \frac{b}{\alpha} \right), \]
\[ a > 0 ; \]
\[ \int_0^r r^{2n+2\eta+1-1} e^{-\alpha t} \sin (b r^2) \, dt = \frac{\Gamma(\alpha + \beta + \lambda)}{(\alpha^2 + b^2)^{\eta+1}} \sin \left( (\alpha + \beta + \lambda) \arctan \frac{b}{\alpha} \right), \]
\[ a > 0 ; \] \( \diamond \)

Theorem 8. Let's the function \( Q(x, y) := \frac{\Gamma(\alpha + \beta + \lambda)}{(\alpha^2 + b^2)^{\eta+1}} \cos \left( (\alpha + \beta + \lambda) \arctan \frac{b}{\alpha} \right) \), \( a > 0 \) satisfy the conditions of function \( Q(x, y) \) of at least one of Theorems 1-4. Then for any \( a > 0, \alpha > 0, \beta > 0, \lambda \geq 0 \) the following formulas will be valid:

\[ \int_0^r \int_0^r \frac{\Gamma(\alpha + \beta + \lambda)}{(\alpha^2 + b^2)^{\eta+1}} \cos \left( (\alpha + \beta + \lambda) \arctan \frac{b}{\alpha} \right) \, dy \, dx = \]
\[ = B_{\eta_1}(\alpha, \beta) \int_0^r \Gamma(\alpha + \beta + \lambda) (x + y) \, dx \, dy = \]
\[ \int_0^r \int_0^r \frac{\Gamma(\alpha + \beta + \lambda)}{(\alpha^2 + b^2)^{\eta+1}} \sin \left( (\alpha + \beta + \lambda) \arctan \frac{b}{\alpha} \right) \, dy \, dx = \]
\[ = B_{\eta_1}(\alpha, \beta) \int_0^r \Gamma(\alpha + \beta + \lambda) (x + y) \, dx \, dy. \]

Proof is similar to Theorem 7. \( \diamond \)

One of the more general possibilities for the practical application of our generalized two-dimensional gamma-function is provided by the following theorem.

Theorem 9. Let us assume that following conditions are met:

1) the function \( Q(x, y) = \frac{\Gamma(\alpha + \beta + \lambda)}{(\alpha^2 + b^2)^{\eta+1}} \cos \left( (\alpha + \beta + \lambda) \arctan \frac{b}{\alpha} \right) \) satisfy the conditions of at least one of Theorems 1-4;

2) The function \( f(x) \) is expanded into a Taylor series
\[ f(x) = \sum_{n=0}^\infty a_n x^n, \quad x \in [0, L] ; \quad L \in N ; \] (42)

3) Monotone sequence of natural numbers \( \{a_n, n \in N\} \), where \( \forall n \in N : a_n > 0 \), satisfies the following condition
\[ \exists M_n \in N \quad \forall n > M_n : na_n < A, \quad A < +\infty . \]

Then for any \( \alpha > 1, \beta > 1, \lambda > 0 \) the following formulas will be valid:

\[ B_{\eta_1}(\alpha, \beta) = B(\alpha, \beta) \frac{\Gamma(\alpha + \beta + \lambda) \sum_{n=0}^\infty \Gamma(\alpha + n) a_n}{\Gamma(\alpha + \beta + n)} \] (43)

\[ \Gamma \left( R^2 ; f \left( \frac{n}{y^n + a} \right) g \left( \frac{\beta + n}{b^n + a} \right) \right) (\alpha, \beta, \lambda) = \int_0^\infty \left( \frac{\Gamma(\alpha + \beta + \lambda) \sum_{n=0}^\infty \Gamma(\alpha + n) a_n}{\Gamma(\alpha + \beta + n)} \right) \left( \frac{\Gamma(\alpha + \beta + \lambda) \sum_{n=0}^\infty \Gamma(\alpha + n) a_n}{\Gamma(\alpha + \beta + n)} \right) \left( \frac{\Gamma(\alpha + \beta + \lambda) \sum_{n=0}^\infty \Gamma(\alpha + n) a_n}{\Gamma(\alpha + \beta + n)} \right) \right) \left( \frac{\Gamma(\alpha + \beta + \lambda) \sum_{n=0}^\infty \Gamma(\alpha + n) a_n}{\Gamma(\alpha + \beta + n)} \right) \]

Proof. Multiplying both sides of equality (42) by \( x^{-\eta} (1 - x)^{-\eta} \), and integrating from 0 to 1, we obtain

\[ B_{\eta_1}(\alpha, \beta) = \sum_{n=0}^\infty a_n B(\alpha + n, \beta) = \Gamma(\beta) \sum_{n=0}^\infty \Gamma(\alpha + n) a_n = B(\alpha, \beta) \frac{\Gamma(\alpha + \beta + \lambda) \sum_{n=0}^\infty \Gamma(\alpha + n) a_n}{\Gamma(\alpha + \beta + n)} . \] (43)
We find the last two formulas using the definition of the two-dimensional generalized gamma function and generalized formula (24) for the relationship between the gamma function and the beta function. ∎

Remark 6. Obviously, all theorems, lemmas and remarks of this section will be valid if we consider the case \( Q(x,y) := \int_{0}^{\infty} \int_{0}^{\infty} g(x+y) \) of the function \( Q(x,y) \). Namely, if we transform case \( Q(x,y) \) of the form \( Q(x,y) = \int_{0}^{\infty} \left( \frac{y}{x+y} \right) g(x+y) \), we can easily extend all the result to this case.

As an additional application, we can consider the hypergeometric function of the form [3]:

\[
F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_{0}^{\infty} \left( \frac{x^{y-b-1}}{(1-x)^{y-c-1}} \right) dx = B_{(y-c), (b, c-b)}, \tag{44}
\]

where \( \text{Re}(c) > \text{Re}(b) > 0 \), \( z \) is not a real number that is greater than or equal to 1.

We can similarly represent this function as a special case of a two-dimensional generalized gamma function, in the form

\[
F(a,b;c;z) = \frac{1}{\Gamma(b)\Gamma(c-b)} \int_{0}^{\infty} \left( \frac{x^{y-b-1}}{(1-x)^{y-c-1}} \right) f(x) dx,
\]

for the same parameters, using the equality (44).

Also, if we consider a similar generalization for the hypergeometric function in the form

\[
F_{\gamma_{1}}(a,b;c;z) = \frac{1}{\Gamma(b)\Gamma(c-b)} \int_{0}^{\infty} \left( \frac{x^{y-b-1}}{(1-x)^{y-c-1}} \right) f(x) dx,
\]

where \( \text{Re}(c) > \text{Re}(b) > 0 \), \( z \) is not a real number greater than or equal to 1, the function \( f(x) \), is such that the integral (45) converges. Let’s the function

\[
Q(x,y) := f \left( \frac{y}{x+y} \right) g(x+y)(x+y(1-x))^{y} (x+y)^{y}.
\]

it satisfy the conditions of function \( Q(x,y) \) of at least one of Theorems 1-4, taking into account the Remark 5. Then we obtain

\[
F_{\gamma_{1}}(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)\Gamma(c+\lambda)} \int_{0}^{\infty} \int_{0}^{\infty} \left( \frac{x^{y-b-1}}{(1-x)^{y-c-1}} \right) f(x) dx
\]

where \( \text{Re}(c) > \text{Re}(b) > 0 \), \( \text{Re}(\lambda) \geq 0 \), \( z \) is not a real number greater than or equal to 1, the function \( g \) identically not equal to zero.

If we assume \( g = 1 \), and the function \( Q(x,y) := f \left( \frac{y}{x+y} \right) (x+y(1-x))^{y} (x+y)^{y} \)

it satisfy the conditions of function \( Q(x,y) \) of at least one of Theorems 1-4, we obtain

\[
F_{\gamma_{1}}(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)\Gamma(c+\lambda)} \int_{0}^{\infty} \int_{0}^{\infty} \left( \frac{x^{y-b-1}}{(1-x)^{y-c-1}} \right) f(x) dx
\]

where \( \text{Re}(c) > \text{Re}(b) > 0 \), \( \text{Re}(\lambda) \geq 0 \), \( z \) is not a real number greater than or equal to 1.

Obviously, formulas (46) and (47) for the generalized hypergeometric function (45) give us many options for choosing function \( g \) and parameter \( \lambda \).

Conclusion. In the article we showed that a special case (23) of the two-dimensional generalized gamma function (3) is the product of the one-dimensional generalized beta function (2) and the one-dimensional generalized beta function (4). Thus, formula (24) for this special case (23) is a generalization of formula for the relationship between the classical gamma function and beta function. We also obtained quite a few properties of this generalization of the two-dimensional gamma function, including a formula for its differentiation of any order. Next, we showed some practical applications of this generalization, including its use for transforming a one-dimensional generalized hypergeometric function into a two-dimensional generalized gamma function.
References: